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NOTES AND QUERIES.

ON THE FORM OF THE NUMBER WHOSE LOGARITHM IS EQUAL TO ITSELF.

THE logarithm of any number is commonly to be characterized as being less than that number, for in all systems zero is the logarithm of unity. When, however, it is asked what is the logarithm of an infinite number, the reply may also be, an infinite number, and thereby the number and logarithm may be considered as virtually resolving into each other. The question of *form*, however, still remains. In illustration of this, the equation $y=\varepsilon^y$ may be taken, thence to find the series at once representative of y in its double character of number and logarithm to itself. To obtain the aid of Lagrange's theorem put $y=a+x\phi y$, and thereupon y becomes equal to

$$a+x\phi a+\frac{x^2}{1.2}(\phi a)^2\frac{d}{da}+\frac{x^3}{1.2.3}(\phi a)^3\frac{d^2}{da^2}+\text{\&c.}$$

Reapplying this to the first equation, a becomes o, $x=1$, and $\phi y=\varepsilon^y$, so that $\phi a=1$, and the different orders of the differential coefficients become 2, 3^2, 4^3; hence

$$y=1+\frac{2}{2}+\frac{3.3}{2.3}+\frac{4.4.4}{2.3.4}+\frac{5.5.5\,5}{2.3.4\,5}, \text{\&c.},$$

the form sought. The hint of the preceding simple resolution has been taken from some papers by that eminent mathematician, the late Rev. Robert Murphy, Esq., M.A., author of *The Theory of Equations, &c.*, among some scraps preserved by a pupil of his (Robert Tucker, Esq.), and kindly sent to the present writer for examination. Among the same scraps there is a simple shaping out of the process to be followed in arriving at Mr. Murphy's own well known theorem, fairly characterized by Professor De Morgan as "one of the most general and interesting contributions which analysis has received for many years." (*Diff. Cal.*, p. 328.)

If α be a root of the equation $\phi x=o$, then $-\alpha$ is the coefficient of $\frac{1}{x}$ in the expansion of log. $\frac{\phi x}{x}$.

For let β, γ, &c., be the other roots, then $\phi x=\mathrm{C}\times(x-\alpha)(x-\beta)(x-\gamma)$, &c., and

$$\frac{\phi x}{x}=\mathrm{C}^1\times\left(1-\frac{\alpha}{x}\right).\left(1-\frac{x}{\beta}\right).\left(1-\frac{x}{\gamma}\right)\text{\&c.};$$

therefore

$$\log.\frac{\phi x}{x}=\log.\,\mathrm{C}^1+\log.\left(1-\frac{\alpha}{x}\right)+\log.\left(1-\frac{x}{\beta}\right)+\text{\&c.},$$

of which the only part which contains negative powers of x is

$$\log.\left(1-\frac{\alpha}{x}\right), \text{ or } -\alpha\frac{1}{x}+\left(\frac{\alpha^2}{2}.\frac{1}{x^2}\right)-\left(\frac{\alpha^3}{3}.\frac{1}{x^3}\right)+\text{\&c.};$$

therefore coefficient of $\frac{1}{r}=-\alpha$.

Although to the general reader the preceding illustrations may appear simply mathematical, yet the student of life assurance and probability will readily detect their application, if he happen to remember, as it has been shown elsewhere, that the Napierian logarithm is representative of what the writer has been tempted to call integral as contrasted with differential mortality.

EDWIN J. FARREN.

We have been favoured by Mr. C. W. Merrifield, of the Privy Council Office, with the following method of obtaining an expression for the rate of interest in immediate annuities:—

Let q be the present value of an annuity, divided by the amount, r the rate of interest, and $n(=m-1)$ the term of years—the annuity being *in possession*, not *deferred.*

By the common formula, $q=\frac{(1+r)^{n-1}}{r(1+r)^n}$; and

$$\therefore \frac{1}{1+r}=\frac{1}{q+1}\left\{q+\left(\frac{1}{1+r}\right)^{n+1}\right\}=\frac{1}{q+1}\left\{q+\left(\frac{1}{1+r}\right)^{m}\right\}.$$

By Laplace's theorem, we know that, if $u=\mathrm{F}(z+y\phi u)$, F, ϕ, and ψ being any functions whatever,

$$\psi u=\psi\mathrm{F}z+\phi\mathrm{F}z\frac{d\psi\mathrm{F}z}{dz}y+\frac{d}{dz}\left\{(\phi\mathrm{F}z)^2\frac{d\psi\mathrm{F}z}{dz}\right\}\frac{y^2}{1.2}+\frac{d^2}{dz^2}\left\{(\phi\mathrm{F}z)\ \frac{d\psi\mathrm{F}z}{dz}\right\}\frac{y}{1.2.3}+\ldots\ldots$$

making $z=q$, $y=1$, $u=\frac{1}{r+1}$, $\mathrm{F}u=\frac{u}{q+1}$, $\phi u=u^m$, $\psi u=\frac{1}{u}=r+1$;

and therefore

$$\phi\mathrm{F}z=\left(\frac{q}{q+1}\right)^m,\ \psi\mathrm{F}z=\frac{q+1}{q}=1+\frac{1}{q},\ \frac{d\psi\mathrm{F}z}{dz}=-\frac{1}{q^2},$$

we have

$$r+1=\frac{q+1}{q}-\frac{q^{m-2}}{(q+1)^m}-\frac{1}{1.2}\frac{d}{dq}\frac{q^{2m-2}}{(q+1)^{2m}}-\frac{1}{1.2.3}\frac{d^2}{dq^2}\frac{q^{3m-2}}{(q+1)^{3m}}-\ldots\ldots\ldots$$

the $(x+1)^m$ term being $-\frac{1}{1.2.3\ldots\ldots x}\frac{d^{x-1}}{dq^{x-1}}\frac{q^{xm-2}}{(q+1)^{xm}}$;

or, performing the differentiations,

$$r=\frac{1}{q}-\frac{q^{m-2}}{(q+1)^m}-\frac{(2m-2)(2m-3)}{1.2}\frac{q^{2m-2}}{(q+1)^{2m}}\left(1-\frac{m}{m-1}\frac{q}{q+1}\right)$$

$$-\frac{(3m-2)\,(3m-3)\,(3m-4)}{1.2.3}\frac{q^{3m-3}}{(q+1)^{3m}}\times$$

$$\left\{1-\frac{2m}{m-1}\frac{q}{q+1}+\frac{3m}{m-1}\frac{3m+1}{3m-2}\frac{q^2}{(q+1)^2}\right\}-\ldots\ldots\ldots\ldots$$

---

* See De Morgan's *Differential and Integral Calculus*, pp. 168-170.

Mr. Merrifield proceeds to show that the convergency of this series depends upon a limit, lying between

$$\left(\frac{nq+q}{nq+n}\right)^{n+1}\frac{(n-q)}{q(q+1)} \text{ and } \left(\frac{nq+q}{nq+n}\right)^{n}\frac{n-q+1}{(q+1)^2},$$

being less than unity, which must always be the case, provided n exceed q, and both n and q exceed unity; whence the series is convergent for every case that can arise in practice. He adds, it is hardly perhaps necessary to remark that, in interpreting these (and other general formulæ for annuities at compound interest) for periods involving a fractional portion of a year, such as $\left(n+\frac{1}{x}\right)$ years, it must be assumed that a payment $= \frac{\text{annuity}}{x}$ is made at the end of the $(n+1)$th year; or, if the fractional payment is made at any other period, it must be (not equal, but) commercially equivalent to $\frac{\text{annuity}}{x}$.

Mr. James Meikle, an abstract of whose paper appears at the commencement of this Number, communicates the following method, which we have slightly abridged, of obtaining the value of a life annuity at one rate of interest from the value at another given rate.

Let $A =$ the given value.

$A' =$ a value required at a lower rate; then

$$A = {}_1av^1+{}_2av^2+{}_3av^3+ \ldots\ldots\ldots\ldots\ldots + {}_nav^n$$

$$A' = {}_1a(v+\Delta v)^1+{}_2a(v+\Delta v)^2+{}_3a(v+\Delta v)^3+\ldots\ldots+{}_na(v+\Delta v)^n$$

$$= {}_1av^1+{}_1a(\Delta v)^1+{}_2av^2+{}_2a2v\Delta v+{}_2a(\Delta v)^2$$

$$+{}_3av^3+{}_3a3v^2\Delta v+{}_3a3v(\Delta v)^2+{}_3a(\Delta v)^3+\&\text{c.}$$

$$= {}_1av^1+{}_2av^2+{}_3av^3+\ldots\ldots\ldots\ldots\ldots\ldots+{}_nav^n$$

$$+\frac{\Delta v}{v}\left\{{}_1av^1+2_2av^2+3_3av^3+4_4av^4+5_5av^5+\ldots\ldots+n_nav^n\right\}$$

$$+\frac{(\Delta v)^2}{v^2}\left\{{}_2av^2+3_3av^3+6_4av^4+10_5av^5+\ldots\ldots\ldots\ldots+\frac{n.n-1}{1.2}{}_nav^n\right\}$$

$$+\&\text{c.}$$

But ${}_1av^1+{}_2av^2+{}_3av^3+\ldots\ldots+{}_nav^n=A$;

and ${}_1av^1+2_2av^2+3_3av^3+\ldots\ldots+n_nav^n$ is the expression for the value of an increasing annuity $=\frac{S_x}{D_x}$.

Also

$${}_2av^2+3_3av^3+6_nav^n+\ldots\ldots\ldots\frac{n.n-1}{1.2}{}_nav^n$$

$$=\frac{S_{x+1}+S_{x+2}+S_{x+3}+\ldots\ldots\ldots S_{x+n}}{D_x}$$

$$=\frac{\Sigma S_{x+1}}{D_x}.$$

In like manner

$$
{}_3av^3+4\,{}_4av^4+10\,{}_5av^5+20\,{}_6av^6+\,\dots\dots+\frac{n\,.\,n-1\,.\,n-2}{1.2.3}\,{}_nav^n
$$

$$
=\frac{S_{x+2}+2S_{x+3}+3S_{x+4}+\dots\dots}{D_x}
$$

$$
=\frac{\Sigma\,S_{x+2}+\Sigma\,S_{x+3}+\Sigma\,S_{x+4}+\dots\dots}{D_x}
$$

$$
=\frac{\Sigma^2\,S_{x+2}}{D_x}:\text{ and so on.}
$$

Hence

$$
A'=A+\frac{\Delta v}{v}\frac{S_x}{D_x}+\frac{(\Delta v)^2}{v^2}\frac{\Sigma S_{x+1}}{D_x}+\frac{(\Delta v)^3}{v^3}\frac{\Sigma^2 S_{x+2}}{D_x}+\&c.
$$

Example.—Let the annuity at age 89, Carlisle 4 per cent., be given. Find the annuity at 89, Carlisle 3 per cent.

$$
\begin{array}{lcr}
A & = & 2\cdot577040 \\
\dfrac{\Delta v}{v}\dfrac{S_x}{D_x} & = & \cdot085865 \\
\dfrac{(\Delta v)^2}{v^2}\dfrac{\Sigma S_{x+1}}{D_x} & = & \cdot001949 \\
\dfrac{(\Delta v)^3}{v^3}\dfrac{\Sigma^2 S_{x+2}}{D_x} & = & \cdot000038 \\
\hline
 & & 2\cdot664892 \\
A' & = & 2\cdot664895
\end{array}
$$

In the same manner it may be shown that the value of a life annuity at a higher rate of interest may be expressed in terms of the value at a lower rate.

$$
A'=A-\frac{\Delta v}{v}\frac{S_x}{D_x}+\frac{(\Delta v)^2}{v^2}\frac{\Sigma\,S_{x+1}}{D_x}-\frac{(\Delta v)^3}{v^3}\frac{\Sigma^2 S_{x+2}}{D_x}+\&c.
$$

* * *

It has been contended, in some recent discussions on questions of probability, that if a coin, admitted to be "dynamically true," be tossed up a hundred times, and turn up head a hundred times successively, the probability at the next throw is in favour of tail; for, argue the supporters of this view, in any given number of trials with such a coin, it is probable that the number of heads turned up will be equal to the number of tails; and as such average must be maintained, the more frequently the one event has occurred the less likely it is to recur thereafter. The fallacy of this argument appears to us to lie in the assumption of an hypothesis, the terms of which are inconsistent. If a coin were "dynamically true," it never would, on being tossed many times in succession, afford the same result. If it did, we should be justified in assuming that it was not of that character, and that some cause was at work which rendered the one event always more probable than the other. Being true, one result might be looked for at any time just as much as the other; and experiment, when of sufficient duration, would no doubt afford evidence that such expectation was well founded.—Ed. *A. M.*